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"Triviality" and the Perturbative Expansion in $\lambda\Phi^4$ Theory

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Abstract:

The "triviality" of $(\lambda\Phi^4)_4$ quantum field theory means that the renormalized coupling λ_R vanishes for infinite cutoff. That result inherently conflicts with the usual perturbative approach, which begins by postulating a *non-zero*, cutoff-independent λ_R. We show how a "trivial" solution $\lambda_R = 0$ can be compatible with the known structure of perturbation theory to arbitrarily high orders, by a simple re-arrangement of the expansion. The "trivial" solution reproduces the result obtained by non-perturbative renormalization of the effective potential. The physical mass is finite, while the renormalized coupling strength vanishes: the two are *not* proportional. The classically scale-invariant $\lambda\Phi^4$ theory coupled to the Standard Model predicts a 2.2 TeV Higgs, but does *not* imply strong interactions in the scalar sector.

1. Suppose we accept that the 4-dimensional $\lambda\Phi^4$ theory is indeed "trivial" [1], meaning that it has no observable particle interactions; what is the theory's effective potential? Since there are no interactions the effective potential can only be the classical potential plus the zero-point energy of the free-field fluctuations. This is the crucial insight of Ref. [2]:— *for a "trivial" theory the one-loop effective potential is effectively exact.* (A recent lattice calculation provides striking confirmation of this fact [3].)

The usual perturbative renormalization [4] is then not appropriate because it would spoil this exactness — it does not properly absorb the infinities, but merely pushes them into "higher-order terms" which are then neglected. However, it is simple to renormalize the one-loop effective potential in an exact way [5, 6, 7, 2]. (This was first discovered in the context of the Gaussian effective potential [8, 9].) The constant background field ϕ, the argument of $V_{\rm eff}$, requires an infinite re-scaling, but the fluctuation field $h(x) \equiv \Phi(x) - \phi$ (i.e., the $p_\mu \neq 0$ projection of the field) is not re-scaled [2]. The particle mass m_h is related to the cutoff Λ and the bare coupling constant $\lambda = \lambda(\Lambda)$ by

$$m_h^2 = \Lambda^2 \exp - \frac{32\pi^2}{3\lambda}. \tag{1}$$

Thus, for m_h to remain finite λ must vanish like $1/\ln(\Lambda/m_h)$ in the continuum limit ($\Lambda \to \infty$). As a consequence one finds that the connected n-point functions at non-zero momentum vanish for $n > 2$, implying no particle interactions; i.e., "triviality". In particular, the connected 4-point function, from which one might have hoped to define a renormalized coupling constant λ_R, vanishes.

The usual perturbative approach, by contrast, is based on an attempt to generate a cutoff-independent and *non-vanishing* λ_R. No meaningful continuum limit is possible in perturbation theory. In fact, as discussed by Shirkov [10], perturbative calculations of the β function up to 5 loops [11] provide the following results: In odd orders, $\beta_{\rm pert}^{1-{\rm loop}}$, $\beta_{\rm pert}^{3-{\rm loop}}$, $\beta_{\rm pert}^{5-{\rm loop}}$ are positive and monotonically increasing. In even orders $\beta_{\rm pert}^{2-{\rm loop}}$, $\beta_{\rm pert}^{4-{\rm loop}}$ each have an ultraviolet fixed point, which would imply a finite bare coupling constant, in contradiction with the rigorous results of Ref. [1]. The magnitude of this spurious fixed point at even orders appears to decrease to zero with increasing perturbative order. A Borel re-summation procedure [10, 11] yields a positive, monotonically increasing β function, as in odd orders. That does not allow a continuum limit because the renormalized coupling will have an unphysical Landau pole.

The moral is that only by abandoning, at the start, the vain attempt to define a non-zero renormalized 4-point function can one obtain a continuum limit. In the effective potential analysis [2] one actually starts from an approximation scheme (one-loop or

Gaussian) in which, by definition, the shifted field $h(x)$ is non-interacting. The resulting effective potential exhibits spontaneous symmetry breaking (SSB) and allows a continuum limit in which "dimensional transmutation" occurs, with massive particles arising from a scale-invariant bare action. The renormalization never introduces a "λ_R" but simply requires the particle mass and V_{eff} to be finite. One finds, as a consequence, that this renormalization implies "triviality" — thereby revealing that the original "approximation" was effectively exact.

In this Letter we shall follow a different route, considering the 4-point function of the already *massive* theory. At the leading-log level, because of the Landau-pole problem, we shall see that the only possibility for defining a continuum limit of the regularized theory corresponds to $\lambda_R = 0$. This yields the same relation (1) as above. We then show that this solution is compatible with all orders of sub-leading logarithms.

2. Let us start by defining λ_R as the 4-point function in the limit of zero external momenta (which for massive particles is not an exceptional point.) We calculate this in terms of the bare, cutoff-dependent coupling $\lambda = \lambda(\Lambda)$, taking into account the basic one-loop bubble of particles with mass m_h. This gives:

$$\lambda_R = \lambda - b_0 \lambda^2 t, \tag{2}$$

where

$$b_0 \equiv \frac{3}{16\pi^2}, \tag{3}$$

$$t \equiv \ln(\Lambda/m_h). \tag{4}$$

It is evident that the actual expansion is not in powers of λ but rather in powers of λ and t. However, one can define a (perturbative) β-function that depends on λ alone:

$$\beta_{\text{pert}} \equiv \Lambda \frac{\partial \lambda}{\partial \Lambda} = \frac{\partial \lambda}{\partial t} = b_0 \lambda^2 + b_1 \lambda^3 + \ldots . \tag{5}$$

[Note that we are defining the β function in terms of the cutoff dependence of the bare coupling constant. In the conventional perturbative context this is completely equivalent to the more usual definition as the renormalization-point dependence of the renormalized coupling constant. Since we want to consider the case where λ_R vanishes identically the above definition is obviously preferable.] Formally, by integrating the β function one resums large logarithms in the series for λ_R/λ: The first term takes into account all leading-log terms, $(b_0\lambda t)^n$; the second term accounts for the sub-leading logarithms $\lambda(b_0\lambda t)^n$, etc.. Using β seemingly allows one to relax the requirement $b_0\lambda t \ll 1$ to just $\lambda \ll 1$.

This is powerful magic, and very familiar, but one should be aware of the hidden assumptions behind it. The $\beta_{\rm pert}$ function is extracted from an RG equation that is satisfied only in a perturbative sense, neglecting higher-order terms. The statement that the leading term is $b_0\lambda^2$ is equivalent to assuming that the leading-log series converges and so can be summed. That is, one is assuming $|b_0\lambda t|< 1$. If the theory were perturbatively aymptotically free this would create no difficulty, but here b_0 is positive and one has the "Landau-pole" problem. Explicitly, the solution to $d\lambda/dt = b_0\lambda^2$, in terms of the boundary condition at $t = 0$, is

$$\lambda(t) = \frac{\lambda(0)}{1 - b_0\lambda(0)t}. \tag{6}$$

One is forced to identify $\lambda_R = \lambda(0)$ for consistency with the original equation (2), which is seen as the first two terms in the infinite expansion of

$$\lambda_R = \lambda(0) = \frac{\lambda(t)}{1 + b_0\lambda(t)t}. \tag{7}$$

One wants to take Λ, and hence t, to infinity, but as t is increased from zero $\lambda(t)$ grows without bound; indeed it becomes infinite at $t = 1/(b_0\lambda_R)$. Thus, the condition $|b_0\lambda t|< 1$ is inevitably violated. No sensible $\Lambda \to \infty$ limit is possible. This pushes the problem of the continuum limit to the next-to-leading level. There, since $b_1 < 0$, one finds an ultraviolet fixed point; but this conflicts with the rigorous results of Ref [1], and in any case it disappears at next-to-next-to-leading order. These results actually signal the inconsistency, in the $\lambda\Phi^4$ case, of assuming that the leading-log series can be naively re-summed.

3. Let us re-examine the β-function approach, relying on just two key ingredients; (i) a basic equation from which one obtains the Λ dependence of λ, and (ii) the necessity of achieving a continuum limit $\Lambda \to \infty$. Our basic equation is Eq. (2) and we attempt to keep λ_R and the physical mass m_h fixed (i.e. Λ independent) while taking the continuum limit $\Lambda \to \infty$. That is, we demand

$$\frac{d\lambda_R}{dt} = 0, \tag{8}$$

which yields

$$\frac{d\lambda(t)}{dt} - b_0\lambda^2(t) - 2b_0 t\lambda(t)\frac{d\lambda(t)}{dt} = 0. \tag{9}$$

In the usual perturbative analysis one would neglect the third term on the left-hand side of the above equation and arrive at

$$\frac{d\lambda(t)}{dt} = b_0\lambda^2(t). \tag{10}$$

Seemingly, the neglected term is then $\mathcal{O}(\lambda(t)^3)$, justifying the procedure, *a posteriori*. However, one cannot obtain a continuum limit in this way, as just explained.

If, instead, we *do* keep the third term in Eq. (9) we obtain

$$\frac{d\lambda(t)}{dt} = -b_0\lambda^3(t)\frac{1}{\lambda(t) - 2\lambda_R}. \tag{11}$$

Assuming that $\lambda(t)$ and λ_R are both non-negative we find that Eq. (11) has to be studied separately for $\lambda(t) - 2\lambda_R > 0$ and for $\lambda(t) - 2\lambda_R < 0$ to preserve the uniqueness of the solution. In neither case, however, is a limit $t \to \infty$ possible if $\lambda_R > 0$. The only possibility is associated with the case $\lambda_R = 0$, which gives:

$$\frac{d\lambda(t)}{dt} = -b_0\lambda^2(t), \tag{12}$$

$$\lambda(t) = \frac{1}{b_0 t}. \tag{13}$$

Thus, now we find a *negative* β function, giving a bare coupling constant that tends to zero in the continuum limit. Eq. (13) is precisely the relation (1), obtained from the effective-potential analysis of the massless theory [2]. [The above explicitly answers the objection of Ref. [12]: our β function is not, of course, β_{pert} + (non-perturbative corrections); it is simply the *right* β function for achieving a continuum limit.]

4. To discuss higher orders it is convenient to introduce the variable

$$x = b_0\lambda(t)t. \tag{14}$$

The basic one-loop correction, Eq. (2), then has the form $\lambda_R^{(0)} = \lambda(t)(1-x)$. Explicit calculation of the higher-order leading-logarithmic corrections to this formula would of course give $\lambda_R = \lambda(t)(1 - x + x^2 - x^3 + \ldots)$, in agreement with a formal expansion of $\lambda_R = \lambda(t)/(1+x)$ (Eq. (7)). However, that expression represents a re-summation of the geometric series that is only valid if $|x| < 1$. Our solution, Eq. (13), is $x = 1$ with $\lambda_R = 0$. It is easy to see that this can be a solution to arbitrarily high order if we rearrange the perturbative expansion suitably. We can view the higher-order diagrams as modifying, and multiplicatively renormalizing, $\lambda_R^{(0)}$ rather than $\lambda(t)$. In a sense, this makes the effective expansion parameter $x^n(1-x)$ rather than x^n itself. For $x \ll 1$ this would make essentially no difference, of course. It produces a sequence of approximations (for $N = 0, 1, 2, \ldots$) of the form

$$\lambda_R^{(N)} = \lambda(t)(1-x)(1 + x^2 + x^4 + ...x^{2N}) = \lambda(t)\frac{1-(x^2)^{N+1}}{1+x} \tag{15}$$

which, for any N, gives

$$\lambda_R^{(N)}\Big|_{x=1} = 0. \tag{16}$$

Note that the limits $x \to 1$ and $N \to \infty$ do not commute. Indeed, for any $x < 1$ one has

$$\lambda_R = \lim_{N\to\infty} \lambda_R^{(N)}(x) = \frac{\lambda(t)}{1+x}, \tag{17}$$

whose $x \to 1$ limit is $\lambda_R = \frac{1}{2}\lambda(t)$, whereas we have

$$\lambda_R = \lim_{N\to\infty} \lim_{x\to 1} \lambda_R^{(N)}(x) = 0, \tag{18}$$

yielding again Eqs. (12, 13).

This procedure can be extended to include all orders of sub-leading logarithms. The essential point is that any sub-leading-log term A appearing at some order in λ will itself be modified in subsequent orders by a series of leading-log corrections, $A(1 - x + x^2 - \ldots)$, and so is multiplied by a $\lambda_R^{(N)}$ factor. For instance, the sequence of approximations

$$\lambda_R^{(N,M+1)} = \frac{\lambda_R^{(N)}}{1 - c\lambda_R^{(N)} \ln \frac{\lambda_R^{(N,M)}(1+c\lambda(t))}{\lambda(t)(1+c\lambda_R^{(N,M)})}}, \tag{19}$$

with $c \equiv b_1/b_0$, contains, in the limit $N \to \infty$, $M \to \infty$, all the leading and next-to-leading corrections to the zero-momentum coupling, λ_R. One can see this as follows. For $|x| < 1$ the above sequence corresponds to an iterative solution of the implicit equation

$$\lambda_R = \frac{\lambda_{ll}}{1 - c\lambda_{ll} \ln \frac{\lambda_R(1+c\lambda(t))}{\lambda(t)(1+c\lambda_R)}} \tag{20}$$

where $\lambda_{ll} = \lim_{N\to\infty} \lambda_R^{(N)}$ is the leading-log solution, which is $\lambda_{ll} = \lambda(t)/(1+x)$ for $|x| < 1$. It is then straightforward to check that for λ_R to be cutoff independent one requires $\lambda(t)$ to satisfy

$$\frac{d\lambda(t)}{dt} = b_0\lambda(t)^2(1 + c\lambda(t)), \tag{21}$$

which is the two-loop perturbative β function. However, for $x = 1$ the sequence (19) gives identically

$$\lambda_R = \lim_{N\to\infty} \lim_{M\to\infty} \lim_{x\to 1} \lambda_R^{(N,M)}(x) = 0. \tag{22}$$

and the associated relations (12, 13). [Note that the re-summations producing the logarithmic term in the denominator of Eq. (19) can be performed consistently even when $x \to 1$ as $t \to \infty$ since both $\lambda_R^{(N)}$ and $\lambda(t)$ vanish in that limit.]

In other words, we have exploited the fact that the structure of the sub-leading logarithms can be inferred from the usual perturbative β function, which just represents a

formal re-summation of those terms. However, that re-summation is valid only for $|x| < 1$. The sub-leading logarithmic structure itself, though, when examined *at* $x = 1$, is consistent to all orders with the 'trivial' solution $\lambda_R = 0$. The point is that all sub-leading corrections are themselves multiplied by a $\lambda_R^{(N)}$ factor.

Of course all of the above is open to the objection that we are merely re-arranging the terms of a divergent series. There is no defence to this charge. Our point, though, is that the conventional procedure, re-summing leading logs to all orders, then sub-leading logs, etc., is itself a re-arrangement of a divergent series. Moreover, because of the Landau pole, one is forced into a region with $x \geq 1$ where this re-arrangement is highly dubious because the sub-series being re-summed are themselves divergent.

5. To further illustrate our point we give a concrete example. This is not meant to represent how things actually work in $\lambda\Phi^4$ theory, but merely to reinforce the point that the conventional procedure, although sanctified by time and custom, can in fact give the wrong answer. Consider the mathematical example in which λ_R and the bare λ are related by:

$$\lambda_R = \lambda(1-x)\sum_{n=0}^{\infty} g_n(\delta)x^{2n}, \tag{23}$$

where δ is a parameter that vanishes in the infinite-cutoff limit (say, as $1/\Lambda$). If the coefficients $g_n(\delta)$ all become unity in the infinite-cutoff limit (i.e., as $\delta \to 0$), then this reproduces the leading-log series $\lambda_R = \lambda(1 - x + x^2 - \ldots)$. However, suppose that in the double limit $\delta \to 0$ and $n \to \infty$

$$g_n(\delta) \to \begin{cases} 1 & \text{if } n\delta \ll 1, \\ 0 & \text{if } n\delta \gg 1. \end{cases} \tag{24}$$

This could happen in many ways; e.g. $g_n(\delta) = (1-\delta)^n$ or $g_n(\delta) = 1/(1 + n!\delta^n)$. While all g_n's become unity as $\delta \to 0$ for any finite n, we must be careful because our series involves infinitely large n. For any finite δ, no matter how small, the g_n coefficients at very large n ($n > 1/\delta$) become much less than unity. Thus, for $\delta \to 0$ we have

$$\begin{aligned} \lambda_R &\sim \lambda(1-x)\left(\sum_{n=0}^{1/\delta-1} x^{2n} + \sum_{n=1/\delta}^{\infty} g_n(\delta)x^{2n}\right) \\ &\sim \lambda(1-x)\left(\frac{(1-x^{2/\delta})}{(1-x^2)} + R(x)\right) \\ &\sim \lambda\frac{(1-x^{2/\delta})}{(1+x)} + \lambda(1-x)R(x), \end{aligned} \tag{25}$$

The remainder term $R(x)$ is a series beginning at order $x^{2/\delta}$ with a radius of convergence greater than unity, and so is non-singular at $x = 1$.

For $|x| < 1$ one has $x^{2/\delta} \to x^{\infty} \to 0$, and $R(x) \to 0$, so that:

$$\lambda_R \sim \frac{\lambda}{(1+x)}, \tag{26}$$

which is the usual perturbative relationship, at the leading-log level. In this case the subtlety about the $\delta \to 0$ limit of the g_n's is irrelevant.

However, for $x = 1$ the last equation is *not* valid. One then has $x^{2/\delta} = 1^{\infty} = 1$ in (25), and so $\lambda_R = 0$. This is obvious from the $(1 - x)$ factor in the original equation, (23). The point is that the $g_n x^{2n}$ series does *not* generate a $1/(1 - x)$ factor to cancel it. Thus, this example admits the "triviality" solution, $\lambda_R = 0$ and $x = 1$, associated with the *negative* β function of Eq. (12).

6. In conclusion, we have presented a simple rearrangement procedure which reproduces the full perturbative expansion at arbitrarily high orders and is valid in the full range $x \leq 1$ ($x \equiv b_0 \lambda t$). It is based on the simple remark that $x^n(1-x)$ is a more suitable expansion parameter than x^n itself. The assumption $x < 1$ allows one to re-sum the various sub-series and leads to the conventional results. However, one cannot then obtain any consistent continuum limit, and moreover one cannot avoid being dragged into a region with $x \geq 1$, invalidating the original assumption. However, if the continuum limit is governed by $x \to 1$, then the condition $\lambda_R = 0$ holds to all orders in this modified expansion. This solution is entirely consistent with the "triviality" found in mathematically rigorous analyses [1]. It is also entirely consistent with the effective-potential analysis [2], which is based on the very physical consideration that the effective potential of a "trivial" theory is just the classical potential plus the zero-point energy of the free-field fluctuations.

Our results here *prove* nothing, since we start from an inherently divergent Feynman-diagram expansion. However, they do provide a way to understand how "triviality" can be consistent with a seemingly highly non-trivial perturbative structure.

The consequences of our picture are substantial, and are discussed in more detail in Ref. [2]. Although the $\lambda\Phi^4$ theory is "trivial" (i.e., has non-interacting particles), it has SSB. When coupled to the Standard Model — and the gauge and Yukawa interactions may be treated as small perturbations — it leads to the Higgs-Kibble mechanism in the usual way. In the theoretically most attractive classically-scale-invariant case, one finds [6, 2] the relation $m_h^2 = 8\pi^2 v^2$, where v is the renormalized expectation value of the scalar field, known from the Fermi constant to be 246 GeV. Thus, one predicts a 2.2 TeV Higgs boson

[6, 2]. In the perturbative picture m_h would be proportional to λ_R, but in the "trivial" solution m_h and λ_R are quite distinct quantities: the former remains finite while the latter vanishes [13]. Thus, in spite of the large Higgs mass, the Higgs/longitudinal-W, Z sector in our picture is *not* strongly interacting. Indeed, the interactions in this sector are of electroweak strength, and would vanish if the gauge and Yukawa couplings were turned off.

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